MMAT5030 Notes 6

1 Uniform Convergence of Series of Functions of Two variables

Results in this section are essential contained in Notes 2. The only difference is now there are two independent variables, which is necessary for partial differential equations.

Let $f_n, n \geq 1$, be functions defined on some rectangle $R = [a, b] \times [c, d]$. We consider the series of functions $\sum_{n=1}^{\infty} f_n$. This series **pointwisely converges** to a function F if for each $(x, y) \in R$, the series of numbers $\sum_{n=1}^{\infty} f_n(x, y)$ converges to the number $F(x, y)$. In other words, for each $(x, y) \in R$ and $\varepsilon > 0$, there is some n_0 depending on (x, y) and ε such that

$$|\sum_{k=1}^{n} f_k(x, y) - F(x, y)| < \varepsilon \ , \quad \forall n \geq n_0.$$

It **uniformly converges** to F if the number n_0 can be chosen independent of (x, y), that is, for $\varepsilon > 0$, there is some n_0 such that

$$|\sum_{k=1}^{n} f_k(x, y) - F(x, y)| < \varepsilon \ , \quad \forall n \geq n_0, \quad \forall (x, y) \in R \ .$$

It is clear that uniform convergence implies pointwise convergence but the converse is not true. Usually we denote the limit function F by $\sum_{n=1}^{\infty} f_n$. Thus $\sum_{n=1}^{\infty} f_n$ has two meanings, first it is the notation for an infinite series, and second, the limit of the infinite series when it converges.

We recall two basic properties of uniform convergence.

Theorem 1 (Continuity Theorem). Suppose that each f_n is continuous on R and the series $\sum_{n=1}^{\infty} f_n$ uniformly converges. Its limit function $\sum_{n=1}^{\infty} f_n$ is continuous on R.

In brief, uniform convergence preserves continuity.

Theorem 2 (Differentiation Theorem). Suppose that (a) the series $\sum_{n=1}^{\infty} f_n$ uniformly converges to F, and (b) all $\partial f_n/\partial x$ and $\partial f_n/\partial y$ exist and the series $\sum_{n=1}^{\infty} \partial f_n/\partial x$ and $\sum_{n=1}^{\infty} \partial f_n/\partial y$ uniformly converge to G_1 and G_2 respectively on R. Then $\partial F/\partial x$ and $\partial F/\partial y$ exist and equal to G_1 and G_2 respectively.

This theorem implies the commutative formula:

$$\frac{\partial}{\partial x} \left(\sum_{n=1}^{\infty} f_n \right) (x, y) = \sum_{n=1}^{\infty} \frac{\partial f_n}{\partial x}(x, y) \ ,$$

and

$$\frac{\partial}{\partial y} \left(\sum_{n=1}^{\infty} f_n \right) (x, y) = \sum_{n=1}^{\infty} \frac{\partial f_n}{\partial y}(x, y) \ ,$$

that is, summation and differentiation are commutative.

Given a series of functions, how can we show that it is uniformly convergent? The most common method is Weierstrass' M-Test.

Theorem 3 (M-Test). Let $\sum_{n=1}^{\infty} f_n$ be a series of functions defined on R. Suppose that there exists $a_n, n \geq 1$, satisfying (a) $|f_n(x,y)| \leq a_n$, for all n and $(x,y) \in R$, and (b) $\sum_{n=1}^{\infty} a_n < \infty$. Then $\sum_{n=1}^{\infty} f_n$ is uniformly convergent.

Example 1. Show that the series $F(x,y) = \sum_{n=1}^{\infty} f_n(x,y)$ is infinitely differentiable provided for each differential operator

$$D = \frac{\partial^N}{\partial x^m \partial y^{N-m}},$$

there are some positive C, k, ρ such that

$$|Df_n(x,y)| \leq Cn^k e^{-n^2\rho} , \quad \forall (x,y) \in R, n \geq 1 .$$

We recall that for $x \geq 0$,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \geq \frac{x^{k+2}}{(k+2)!} .$$

(Just keep the $(k+2)$-th term). Therefore,

$$|Df_n(x,y)| \leq Cn^k e^{-n^2\rho} \leq C(k+2)!\rho^{-k-2}\frac{1}{n^{k+2}} \equiv a_n .$$

Clearly, $\sum_{n=1}^{\infty} a_n = const. \sum_{n=1}^{\infty} 1/n^{k+2} < \infty$. It follows from a repeated application of Theorem 2 and the M-Test, F is differentiable up to any order.

2 The Initial-Boundary Value Problem for the Heat Equation

We first consider the initial-boundary value problem for the one dimensional heat equation under the Dirichlet boundary condition

$$\begin{cases} u_t = u_{xx} & \text{in } [0,\pi] \times (0,\infty) , \\ u(x,0) = f(x) & \text{in } [0,\pi], \\ u(x,t) = 0 & \text{at } x = 0,\ \pi \text{ and } t > 0, \end{cases} \tag{1}$$

where, for consistency, $f(x) = 0$ at $x = 0$ and π. The physical meaning is, $f(x)$ represents the initial distribution of temperature along a bar of length π. With its ends keeping at zero degree all the time, we would like to determine the temperature of the bar at the position-time which is given by the function $u(x,t)$. We have normalized the diffusion coefficient to 1 and the length of bar to π. The general case can be reduced to this normalized one by a scaling, see below. See Text or Wiki under "heat equation" for the derivation of the heat equation and background.

To apply the ideas from Fourier series, we may extend the solution to $[-\pi,\pi]$ as an odd function by setting

$$U(x,t) = -u(-x,t), \quad x \in [-\pi,0) .$$

Observing that $U(-\pi,t) = U(\pi,t) = u(\pi,t) = 0$, we see that the 2π-periodic extension of $U(\cdot,t)$ is a continuous, piecewise smooth function when u is of the same type. Consequently, its Fourier series (which is a sine series) converges uniformly to $U(\cdot,t)$ for each t:

$$U(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin nx .$$

To proceed formally, in order this sine series represents a solution to (1), we require

$$u_t - u_{xx} = \sum_{n=1}^{\infty} (b_n'(t) + n^2 b_n(t)) \sin nx = 0.$$

At $t = 0$,

$$u(x, 0) = \sum_{n=1}^{\infty} b_n(0) \sin nx.$$

On the other hand, let

$$f(x) = \sum_{n=1}^{\infty} B_n \sin nx.$$

(Using $f(0) = f(\pi) = 0$, one can extend f to be an odd function and therefore it admits a sine series representation.) By comparing with the sine series of u, we see that it is necessary to take $b_n(0) = B_n$. Thus we need to solve

$$b_n'(t) + n^2 b_n(t) = 0, \qquad b_n(0) = B_n.$$

This is a standard linear ODE whose solution is given by

$$b_n(t) = B_n e^{-n^2 t} .$$

Putting $b_n(t)$ back to the sine series for $u(x, t)$, we get a formal solution to (1)

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin nx. \tag{2}$$

The boundary condition is formally satisfied as each $\sin nx$ vanishes at 0 and π. To show that (2) really defines a solution, we need to prove that this series converges to some sufficiently regular function. This can be achieved under a very mild regularity assumption on the initial function f.

Theorem 4 *Consider* (1) *where* $f \in R[0, \pi]$. *The series*

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin nx \tag{3}$$

where B_n *is given by*

$$f(x) = \sum_{n=1}^{\infty} B_n \sin nx,$$

or equivalently,

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx,$$

defines a function u *which is infinitely differentiable in* $[0, \pi] \times (0, \infty)$ *and solves* (1). It also satisfies $u(0, t) = u(\pi, t) = 0$ for $t > 0$.

Proof. Consider the series $\sum_{n=1}^{\infty} f_n$ where $f_n(x, t) = B_n e^{-n^2 t} \sin nx$. The partial derivatives of f_n are of the form $B_n n^k e^{-n^2 t} \sin nx$ or $B_n n^k e^{-n^2 t} \cos nx$ for some k depending on the order of differentiation. As f is integrable and B_n is the Fourier coefficients of f, there is some M such that $|B_n| \leq M$ for all n. Therefore, for all $(x, t) \in [0, \pi] \times [t_0, \infty)$ for a fixed $t_0 > 0$, we have the estimate

$$|f_n(x, t| \leq |B_n n^k e^{-n^2 t} \sin nx| \leq M n^k e^{-n^2 t_0} ,$$

($\sin nx$ or replaced by $\cos nx$). It follows from Example 1 (taking $\rho = t_0$) that the function defined by $u(x,t) = \sum_{n=1}^\infty B_n e^{-n^2 t} \sin nx$ is infinitely many times differentiable on $[0,\pi] \times [t_0, \infty)$ for any $t_0 > 0$. Also, for each fixed $(x,t), t > 0$, by Theorem 3,

$$u_x(x,t) = \sum_n nB_n e^{-n^2 t} \cos nx \ ,$$

$$u_{xx}(x,t) = \sum_n (-n^2) B_n e^{-n^2 t} \sin nx \ ,$$

and

$$u_t(x,t) = \sum_n (-n^2) B_n e^{-n^2 t} \sin nx.$$

We conclude that $u_t = u_{xx}$ holds in $[0,\pi] \times (0,\infty)$. Clearly it also satisfies the boundary conditions.

To show that the initial value is achieved by the solution, we need to impose more regularity on the initial function.

Theorem 5. Let f be a continuous, piecewise smooth function satisfying $f(0) = f(\pi) = 0$ on $[0,\pi]$. Then the solution of (1) constructed in Theorem 4 satisfies

$$\lim_{t\to 0} u(x,t) = f(x) \ , \quad \forall x \in [0,\pi].$$

Proof. Consider again the terms $f_n(x,t) = B_n e^{-n^2 t} \sin nx$. If we can show that $\sum |B_n| < \infty$, in view of

$$|f_n(x,t)| \le |B_n| e^{-n^2 t} | \sin nx| \le |B_n|,$$

we can use $\sum |B_n|$ as a majorant to show that $u(x,t) = \sum_{n=1}^\infty f_n = \sum_{n=1}^\infty B_n e^{-n^2 t} \sin nx$ is uniformly convergent on $[0,\pi] \times [0,\infty)$ and hence it is continuous there. Using $u(x,0) = \sum_{n=1}^\infty B_n \sin nx$, we conclude $\lim_{t\to 0} u(x,t) = u(x,0) = f(x)$.

It remains to show $\sum |B_n| < \infty$. By

$$\sum |B_n| = \sum \frac{1}{n} \times n|B_n| \le \frac{1}{2} \left(\sum \frac{1}{n^2} + \sum n^2 B_n^2 \right) ,$$

it suffices to verify $\sum n^2 B_n^2 < \infty$. As f is piecewise smooth, f' is piecewise continuous and hence integrable. By Bessel's inequality and the fact that the Fourier coefficients of f' are equal to nB_n,

$$\sum n^2 B_n^2 \le \frac{2}{\pi} \int_0^\pi f'^2(x)\, dx < \infty,$$

done.

For the general case, consider

$$\begin{cases} u_t = \kappa u_{xx} & \text{in } [0,l] \times (0,\infty) \ , \\ u(x,0) = f(x) & \text{in } [0,l], \\ u(x,t) = 0 & \text{at } x = 0,\ l \text{ and } t > 0, \end{cases} \qquad (4)$$

Observe that $u(x,t)$ solves (4) if and only if the function $\widetilde{u}(x,t) = u(lx/\pi, l^2 t/\kappa\pi^2)$ solve (1). From this we combine Theorems 4 and 5 to get

Theorem 6 Consider (4) where f is continuous, piecewise smooth and vanishes at endpoints. The series

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\frac{n\pi}{l})^2 \kappa t} \sin \frac{n\pi x}{l},$$

where B_n is given by

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l},$$

or equivalently,

$$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx,$$

defines a function u continuous in $[0,\pi] \times [0,\infty)$ and infinitely differentiable in $[0,\pi] \times (0,\infty)$ which solves (4).

We draw two interesting properties from these theorems. First, the solution $u(x,t) \to 0$ as $t \to \infty$, that is, the temperature eventually dies down to zero. Second, the solution has an instant smoothing feature. Our given initial function is piecewise smooth and continuous, but the solution $u(x,t)$ becomes infinitely many times differentiable for every $t > 0$. This is easily seen from the representation formula where the coefficients $B_n e^{-n^2 t}$ are rapidly decreasing. This reflects the heat distribution is a diffusion process where an instant average-out effect takes place.